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Coxeter elements and root bases

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ABSTRACT

Let $\mathfrak{g}$ be a Lie algebra of type A, D, E with fixed Cartan subalgebra $\mathfrak{h}$, root system R and Weyl group W. We show that a choice of Coxeter element $C \in W$ gives a root basis for $\mathfrak{g}$. Moreover, using the results of Kirillov and Thind (2010) [KT] we show that this root basis gives a purely combinatorial construction of $\mathfrak{g}$, where root vectors correspond to vertices of a certain quiver $\widehat{\Gamma}$, and with respect to this basis the structure constants of the Lie bracket are given by paths in $\widehat{\Gamma}$. This construction is then related to the constructions of Ringel and Peng and Xiao.

1. Introduction

Let Γ be a Dynkin graph of type A, D, E. Let $\mathfrak{g}$ and $U_q(\mathfrak{g})$ be the corresponding Lie algebra and quantum group respectively. By choosing an orientation Ω of Γ, one obtains a quiver $\vec{\Gamma} = (\Gamma, \Omega)$. Ringel used the category $\mathrm{Rep}(\vec{\Gamma})$ of representations of $\vec{\Gamma}$ to realize $\mathfrak{n}_+$ and $U_q\mathfrak{n}_+$ (see [R1,R2]). Peng and Xiao then used a related category, $\mathcal{D}^b \mathrm{Rep}(\vec{\Gamma})/T^2$, to realize the whole Lie algebra $\mathfrak{g}$. The drawback of these constructions is the necessity of choosing an orientation of the Dynkin diagram.

Motivated by these results and the ideas of Ocneanu [O], the main goal of this paper is to use a Coxeter element, and the results in [KT], to construct a root basis in the Lie algebra $\mathfrak{g}$ and to determine the structure constants of the Lie bracket in purely combinatorial terms.

In [KT] it was shown a choice of Coxeter element gives a bijection between R and a certain quiver $\widehat{\Gamma}$, which identifies roots in R and vertices in $\widehat{\Gamma}$. This bijection then identifies vertices in $\widehat{\Gamma}$ with basis vectors E_α. Using this identification and choice of basis, it is possible to determine the structure constants of the Lie bracket from paths in $\widehat{\Gamma}$. Thus it is possible to realize the Lie algebra $\mathfrak{g}$ completely in terms of the quiver $\widehat{\Gamma}$. This construction is then independent of any choice of orientation of Γ or choice of simple roots.

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The case of $U_q(\mathfrak{g})$ for $q \neq 1$ is also of interest. However, a full analysis is the subject of ongoing research.

The main result will now be stated. The proof of this theorem will be left to Section 4 and Section 5. In Section 5 this construction will be related to the constructions of Ringel and Peng–Xiao.

Theorem 1.1. *Let $\mathfrak{g}$ be a Lie algebra of type A, D, E with fixed Cartan subalgebra $\mathfrak{h}$. This gives a root system R with Weyl group W. Fix a Coxeter element $C \in W$.*

(1) *The choice of a Coxeter element C gives a root basis $\{E_\alpha\}_{\alpha \in R}$ for $\mathfrak{g}$.*
(2) *Let $\langle \cdot, \cdot \rangle$ be the de-symmetrization of the bilinear form $(\cdot, \cdot)$ given by*

$$\langle x, y \rangle = \big((1 - C)^{-1} x, y\big)$$

Then the Lie bracket is given by

$$[E_\alpha, E_\beta] = \begin{cases} (-1)^{\langle \alpha, \beta \rangle} E_{\alpha+\beta} & \text{for } \alpha + \beta \in R \\ 0 & \text{for } \alpha + \beta \notin R \text{ and } \alpha \neq -\beta \end{cases}$$

2. Preliminaries

2.1. Notation

Let $\mathfrak{g}$ be a simple Lie algebra of type A, D, E, and let $\mathfrak{h}$ be a fixed Cartan subalgebra. Denote by R the root system, W the Weyl group, and Γ the Dynkin diagram associated to the pair $(\mathfrak{g}, \mathfrak{h})$. Thus Γ is a Dynkin diagram of type A, D, E.

Let $\Pi = \{\alpha_i\}_{i \in \Gamma}$ denote a set of simple roots.

Since the Weyl group acts simply-transitively on sets of simple roots, there is a unique element which takes Π to $-\Pi$. This element is called the longest element and denoted by w_0.

For $i \in \Gamma$ define $i^\vee$ by $-\alpha_{i^\vee} = w_0(\alpha_i)$, where $w_0 \in W$ is the longest element.

A set of simple roots Π is compatible with a Coxeter element $C \in W$ if there is a reduced expression $C = s_{i_1} s_{i_2} \cdots s_{i_r}$, where each simple reflection appears exactly once. In other words, Π is compatible with C if $l^\Pi(C) = r$, where l^Π is the length of a reduced expression in terms of the simple reflections s_i.

Let $U_q(\mathfrak{g})$ be the corresponding quantum group. It is generated by elements E_i, F_i, $K_i^{\pm 1}$, where $i \in \Gamma$. In particular, for $q = 1$ this gives the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

2.2. The quiver $\widehat{\Gamma}$

Given the Dynkin diagram Γ of type A, D, E with Coxeter number h, construct a quiver $\widehat{\Gamma} \subset \Gamma \times \mathbb{Z}_{2h}$ as follows:

(1) Choose a "parity" function $p : \Gamma \to \mathbb{Z}_2$, so that $p(i) = p(j) + 1$ for i, j connected in Γ.
(2) Using p, define the vertex set of $\widehat{\Gamma}$ to be $\widehat{\Gamma}_0 = \{(i, n) \mid p(i) + n \equiv 0 \bmod 2\}$.
(3) The arrows are given by $(i, n) \to (j, n + 1)$ for i, j connected in Γ.
(4) Define a "twist" map $\tau : \widehat{\Gamma} \to \widehat{\Gamma}$ by $\tau(i, n) = (i, n + 2)$.

Example 2.1. For the graph $\Gamma = D_5$ the quiver $\widehat{\Gamma}$ is shown in Fig. 1. Note that this is the Auslander–Reiten quiver of the category $\mathcal{D}^b(\vec{\Gamma})/T^2$ for any choice of orientation Ω. Here the $\mathbb{Z}_{2h}$ direction is vertical and the translation acts vertically, while in most of the literature the $\mathbb{Z}_{2h}$ direction is horizontal and the translation acts horizontally to the right.

A function $h : \Gamma \to \mathbb{Z}_{2h}$ such that $h(i) = h(j) \pm 1$ for i, j connected in Γ will be called a "height function". Note that such a map defines an orientation on Γ by $i \to j$ if i, j are connected and

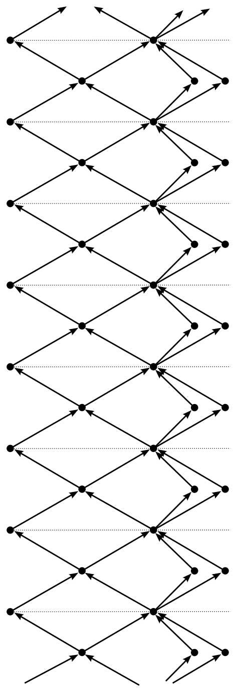

Fig. 1. The quiver $\widehat{\Gamma}$ for graph $\Gamma = D_5$. Recall that $\widehat{\Gamma}$ is periodic, so the arrows leaving the top level are the same as the incoming arrows at the bottom level.

$h(j) = h(i) + 1$. This orientation will be denoted by Ω_h. A height function h also gives an embedding of the quiver (Γ, Ω_h) in $\widehat{\Gamma}$, given by $i \mapsto (i, h(i))$. The image of such an embedding is called a "slice" and is denoted by Γ_h.

For a height function h, if $i \in \Gamma$ is a sink or source for Ω_h define a new height function $s_i h$ by

$$s_i h(j) = \begin{cases} h(i) \pm 2 & \text{if } j = i \text{ where the sign is } + \text{ for } i \text{ a source, } - \text{ for } i \text{ a sink} \\ h(j) & \text{if } j \neq i \end{cases}$$

The orientation determined by $s_i h$ is denoted by $s_i \Omega$ and is obtained by reversing all arrows at i.

Define a function $\langle \cdot, \cdot \rangle_{\widehat{\Gamma}} : \widehat{\Gamma} \times \widehat{\Gamma} \to \mathbb{Z}$ by setting

$$\begin{aligned} \big\langle (i,n), (j,n) \big\rangle_{\widehat{\Gamma}} &= \delta_{ij} \\ \big\langle (i,n), (j,n+1) \big\rangle_{\widehat{\Gamma}} &= \text{the number of paths } (i,n) \to \cdots \to (j,n+1) \\ &= \text{the number of edges between } i, j \text{ in } \Gamma \end{aligned}$$

Then for any $q = (k,m) \in \widehat{\Gamma}$ use the relation

$$\big\langle q, (i,n) \big\rangle_{\widehat{\Gamma}} - \sum_{j-i} \big\langle q, (j,n+1) \big\rangle_{\widehat{\Gamma}} + \big\langle q, (i,n+2) \big\rangle_{\widehat{\Gamma}} = 0$$

for i, j connected in Γ, to extend the definition.

It was shown in [KT, Proposition 7.4] that this function is well defined.

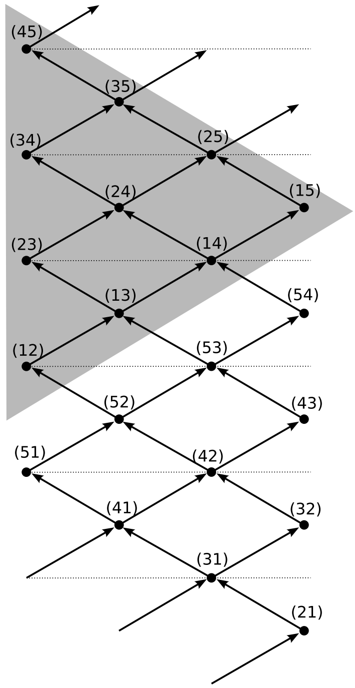

Fig. 2. The bijection $\Phi : R \to \widehat{\Gamma}$ for $\Gamma = A_4$. For each vertex in $\widehat{\Gamma}$ the corresponding root $\alpha \in R$ is shown. The notation (ij) stands for $e_i - e_j$. The set of positive roots corresponding to Π is shaded. Recall that $\widehat{\Gamma}$ is periodic, so that arrows leaving the top level are identified with the incoming arrows on the bottom level.

Given a Coxeter element $C \in W$, it was shown in [KT] that there is a bijection $R \to \widehat{\Gamma}$ with the following properties:

(1) It identifies the Coxeter element C with the "twist" $\tau : \widehat{\Gamma} \to \widehat{\Gamma}$.
(2) It gives a bijection between simple systems Π, compatible with C, and height functions $h : \Gamma \to \widehat{\Gamma}$.
(3) For each height function h one obtains an explicit description of the corresponding positive roots and negative roots as disjoint connected subquivers of $\widehat{\Gamma}$, as well as a reduced expression for the longest element w_0 in the Weyl group. The reduced expression for w_0 is given as a sequence of source to sink reflections taking the slice Γ_{h^Π} to the slice $\Gamma_{h^{-\Pi}}$.
(4) There is a de-symmetrization of the inner product on R, denoted by $\langle\cdot,\cdot\rangle$, which is analogous to the Euler form in quiver theory. Moreover, under the bijection Φ, this form is identified with $\langle\cdot,\cdot\rangle_{\widehat{\Gamma}}$ in $\widehat{\Gamma}$.

Example 2.2. For $\Gamma = A_4$ with $\Pi = \{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5\}$ and $C = s_1 s_2 s_3 s_4$ the bijection $R \to \widehat{\Gamma}$ is given in Fig. 2.

3. Braid group action

In this section the definition and relevant results of the braid group operators as defined in [J] are reviewed. For more details see [J], or [L].

Fix a system of simple roots Π. Let $E_i, F_i, K_i^{\pm 1}$ denote the corresponding generators of $U_q(\mathfrak{g})$.

For simple roots α_i define operators T_i, T_i' on any finite dimensional module V by setting for $v \in V_\lambda$:

$$T_i(v) = \sum_{a,b,c \geqslant 0;\, -a+c-b=m} (-1)^b q^{b-ac} \frac{E_i^{(a)}}{[a]!} \frac{F_i^{(b)}}{[b]!} \frac{E_i^{(c)}}{[c]!} v$$

$$T_i'(v) = \sum_{a,b,c \geqslant 0;\, -a+c-b=m} (-1)^b q^{ac-b} \frac{E_i^{(a)}}{[a]!} \frac{F_i^{(b)}}{[b]!} \frac{E_i^{(c)}}{[c]!} v$$

with $m = (\lambda, \alpha_i\check{})$.

Then there are unique automorphisms of $U_q(\mathfrak{g})$, also denoted by T_i, T_i' so that for any $u \in U_q(\mathfrak{g})$ and $v \in V$ we have $T_i(uv) = T_i(u)T_i(v)$. The operator T_i acts on weights by the reflection s_i.

The automorphisms T_i satisfy the braid relations:

$$T_iT_j = T_jT_i \quad \text{for } (\alpha_i, \alpha_j) = 0$$

$$T_iT_jT_i = T_jT_iT_j \quad \text{for } (\alpha_i, \alpha_j) = -1$$

For the automorphism T_i there are the following formulae:

$$T_iE_i = -F_iK_i$$

$$T_iF_i = -K_i^{-1}E_i$$

$$T_iE_j = E_j \quad \text{for } (\alpha_i, \alpha_j) = 0$$

$$T_iE_j = E_iE_j - q^{-1}E_jE_i \quad \text{for } (\alpha_i, \alpha_j) = -1$$

$$T_iF_j = F_j \quad \text{for } (\alpha_i, \alpha_j) = 0$$

$$T_iF_j = F_iF_j - q^{-1}F_jF_i \quad \text{for } (\alpha_i, \alpha_j) = -1$$

In fact, there are automorphisms T_α for any root α. As above, define T_α on a module V by setting for $v \in V_\lambda$:

$$T_i(v) = \sum_{a,b,c \geqslant 0;\, -a+c-b=m} (-1)^b q^{b-ac} \frac{E_\alpha^{(a)}}{[a]!} \frac{F_\alpha^{(b)}}{[b]!} \frac{E_\alpha^{(c)}}{[c]!} v$$

$$T_i'(v) = \sum_{a,b,c \geqslant 0;\, -a+c-b=m} (-1)^b q^{ac-b} \frac{E_\alpha^{(a)}}{[a]!} \frac{F_\alpha^{(b)}}{[b]!} \frac{E_\alpha^{(c)}}{[c]!} v$$

where $E_\alpha, F_\alpha \in U_q(\mathfrak{g})$ satisfy the $U_q(sl_2)$ relations and $m = (\lambda, \alpha_i\check{})$.

Lemma 3.1. *Let Φ be an automorphism of $U_q(\mathfrak{g})$ such that $E_\alpha = \Phi(E_i)$ and $F_\alpha = \Phi(F_i)$. Then $T_\alpha = \Phi T_i \Phi^{-1}$.*

The automorphisms T_α satisfy relations similar to those of the T_i:

$$T_\alpha E_\alpha = -F_\alpha K_\alpha$$

$$T_\alpha F_\alpha = -K_\alpha^{-1}E_\alpha$$

$$T_\alpha E_\beta = E_\beta \quad \text{for } (\alpha, \beta) = 0$$

$$T_\alpha E_\beta = E_\alpha E_\beta - q^{-1}E_\beta E_\alpha \quad \text{for } (\alpha, \beta) = -1$$

$$T_\alpha F_\beta = F_\beta \quad \text{for } (\alpha, \beta) = 0$$
$$T_\alpha F_\beta = F_\alpha F_\beta - q^{-1} F_\beta F_\alpha \quad \text{for } (\alpha, \beta) = -1$$

Since the operators T_i satisfy the braid relations it is possible to define an operator T_w for any $w \in W$. For any reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ for $w \in W$ define $T_w = T_{i_1} T_{i_2} \cdots T_{i_k}$.

The following lemma will be useful. It can be found in [J, Proposition 8.20].

Lemma 3.2. *If $w \in W$ is such that $w(\alpha_i) \in R_+$, then $T_w(E_i) \in U_q^+$. If, in addition, $w(\alpha_i) = \alpha_j$, then $T_w(E_i) = E_j$.*

For the case to be considered in the following sections, this result gives the following important corollary.

Corollary 3.3. *Let $w_0 \in W$ be the longest element. Then $T_{w_0}(E_{i^\vee}) = T_i E_i = -F_i K_i$.*

Proof. Let $w_0 = s_i w$ be a reduced expression for w_0, so that $T_{w_0} = T_i T_w$. Then since

$$w(\alpha_{i^\vee}) = s_i w_0(\alpha_{i^\vee}) = s_i(-\alpha_i) = \alpha_i$$

the lemma gives $T_w(E_{i^\vee}) = E_i$, and the result follows by applying T_i. $\square$

4. Longest element and construction of root vectors

Let Π be a simple system, and let $R = R_+ \cup R_-$ be the corresponding polarization. Let w_0 be the longest element. A reduced expression $w_0 = s_{i_1} s_{i_2} \cdots s_{i_l}$ is said to be *adapted* to an orientation Ω of Γ if i_k is a source for $s_{i_1} \cdots s_{i_{k-1}} \Omega$. In particular i_1 is a source of Ω.

Lemma 4.1. *Given any orientation Ω, there is a reduced expression adapted to Ω, and moreover, any two expressions adapted to Ω are related by $s_i s_j = s_j s_i$ with $n_{ij} = 0$.*

Proof. Recall that any height function h determines an orientation Ω_h and that for any orientation Ω there is a choice of h so that $\Omega = \Omega_h$. So take some h corresponding to Ω. Note that any reduced expression adapted to Ω gives a sequence of source to sink moves taking the slice Γ_h to the slice Γ_{-h} where Γ_{-h} is the slice corresponding to the simple roots $-\Pi$.

Let $w_0 = s_{i_1} \cdots s_{i_l}$ and $w_0 = s_{i'_1} \cdots s_{i'_l}$ be two different reduced expressions adapted to Ω. Let k be the first index where they differ. Write $i_k = i$ and $i'_k = j$ to simplify notation. Then there are reduced expressions

$$w_0 = w s_i w_1 s_j w_2$$
$$w_0 = w s_j w'_1 s_j w'_2$$

where s_j does not appear in w_1 and s_i does not appear in w'_1. Thus i, j are both sources for $w\Omega$ and hence $n_{ij} = 0$. Note that since w_1 is obtained as a sequence of source to sink reflections, and since s_j does not appear in w_1, j remains a source during this process. Hence if s_k appears in w_1 then k is not adjacent to j, so that $n_{jk} = 0$. Thus $w_1 s_j = s_j w_1$, which gives:

$$\begin{aligned} w_0 &= w s_i w_1 s_j w_2 \\ &= w s_i s_j w_1 w_2 \\ &= w s_j s_i w_1 w_2 \end{aligned}$$

So it is possible to make the two reduced expressions agree at the index k using only the relation $s_i s_k = s_k s_i$ for $n_{ik} = 0$. Continuing in this fashion it is possible to make the expressions agree at every index using only this relation. $\square$

It is well known that a reduced expression $w_0 = s_{i_1} \cdots s_{i_l}$, adapted to Ω, gives an ordering of the positive roots $R = \{\gamma_1, \ldots, \gamma_l\}$ by setting $\gamma_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_k)$. Such an expression also gives root vectors E_α, F_α for $\alpha \in R_+$ as follows:

$$E_{\gamma_k} = T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}) \tag{4.1}$$

$$F_{\gamma_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}) \tag{4.2}$$

Note that since the T_i satisfy the braid relation, Lemma 4.1 implies that the root vectors defined this way do not depend on the choice of reduced expression adapted to Ω.

Note that if $\gamma_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$ then $\gamma_k = s_{\gamma_{k-1}} \cdots s_{\gamma_1} \alpha_{i_k}$, and the longest element can be expressed as $w_0 = s_{\gamma_l} \cdots s_{\gamma_1}$.

Since $T_{\gamma_k} = (T_{i_1} \cdots T_{i_{k-1}}) T_{i_k} (T_{i_1} \cdots T_{i_{k-1}})^{-1}$, then as for reflections,

$$T_{i_1} \cdots T_{i_k} = T_{\gamma_k} \cdots T_{\gamma_1}$$

so the root vectors E_{γ_k} given in Eq. (4.1) can be expressed as

$$E_{\gamma_k} = T_{\gamma_{k-1}} \cdots T_{\gamma_1} E_{i_k} \tag{4.3}$$

Definition 4.2. Let Π be a set of simple roots, and Ω be an orientation of Γ and let $w_0 = s_{i_1} \cdots s_{i_l}$ be a reduced expression adapted to Ω. A root basis $\{E_\alpha\}_{\alpha \in R}$ is said to be adapted to the pair (Π, Ω) if for $\alpha \in R_+$ the vector E_α is given by Eq. (4.1), or equivalently by Eq. (4.3).

4.1. Change of orientation

For a reduced expression $w_0 = s_{i_1} s_{i_2} \cdots s_{i_l}$, adapted to Ω, define a new reduced expression $w_0 = s_{i_2} \cdots s_{i_l} s_{i_1^\vee}$ which is adapted to $s_1 \Omega$. Then this gives a new enumeration of positive roots $\{\gamma_1', \ldots, \gamma_l'\}$, and a new collection of root vectors:

$$\gamma_1' = s_{i_i}(\gamma_{i_2}), \quad \gamma_2' = s_{i_1}(\gamma_{i_3}), \quad \ldots, \quad \gamma_l = \alpha_{i_1^\vee}$$
$$E_{\gamma'} = T_i^{-1}(E_{s_i \gamma}) \quad \text{for } \gamma \neq \alpha_{i_1} \tag{4.4}$$

4.2. Coxeter element

Now consider the case where there is a fixed Coxeter element $C \in W$ and hence an identification $R \to \widehat{\Gamma}$ as in [KT]. In this case a choice of height function h is identified with a set of simple roots Π compatible with C, and hence determines a polarization $R = R_+^h \cup R_-^h$. A height function also determines a reduced expression for w_0 adapted to the orientation Ω_h. This expression is obtained from $\widehat{\Gamma}$ as a sequence of source to sink reflections which take the slice Γ_{h^Π} to the slice $\Gamma_{h^{-\Pi}}$.

Using this reduced expression, there is an associated ordering of the positive roots which gives a completion of the partial order given by paths in $\widehat{\Gamma}$. Note that the completion depends on the reduced expression.

Now choose a height function h. Then using the reduced expression for w_0 obtained above, it is possible to define a collection of root vectors E_α for $\alpha \in R_+^h$ using Eq. (4.1).

Under the identification $R \to \widehat{\Gamma}$ suppose that $\alpha = (i, n)$, then $C\alpha = (i, n+2)$. For j connected to i, denote by γ_j the root corresponding to vertex $(j, n+1)$. The collection of roots $\{\alpha, \gamma_j, C\alpha\}$ is said to satisfy the fundamental relation in $\widehat{\Gamma}$. Such a collection is depicted in Fig. 3.

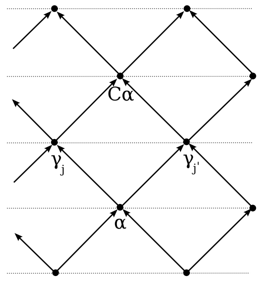

Fig. 3. A collection of roots $\alpha, \gamma_j, C\alpha \in \widehat{\Gamma}$ satisfying the fundamental relation.

Lemma 4.3. *Let $\alpha, \gamma_j, C(\alpha) \in R_+^h$ satisfy the fundamental relation in $\widehat{\Gamma}$. Then the corresponding root vectors satisfy*:

$$E_{C(\alpha)} = \left(\prod_{j-i} T_{\gamma_j}\right) T_\alpha(E_\alpha) \tag{4.5}$$

Proof. Let h be a fixed height function and let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ denote the corresponding set of simple roots and s_i the corresponding simple reflections.

Let $\alpha = (i, n)$, $\gamma_j = (j, n+1)$, $C\alpha = (i, n+2)$ and

$$w_0 = w s_i \left(\prod_{j-i} s_j\right) s_i w'$$

be a reduced expression adapted to Ω_h.

Then

$$\begin{aligned} E_\alpha &= T_w E_{\alpha_i} \quad \text{and} \quad T_\alpha = T_w T_{\alpha_i} T_w^{-1} \\ E_{\gamma_j} &= T_w T_{\alpha_j} E_{\alpha_j} \quad \text{and} \quad T_{\gamma_j} = T_w T_{\alpha_i} T_{\alpha_j} T_{\alpha_i}^{-1} T_w^{-1} \\ E_{C\alpha} &= T_w T_{\alpha_i} \left(\prod_{j-i} T_{\alpha_j}\right) E_{\alpha_i} \end{aligned}$$

where the product $\prod_{j-i}$ is taken over all j connected to i in Γ.

On the other hand, using the first two formulae, and comparing with the third one obtains:

$$\begin{aligned} \left(\prod_{j-i} T_{\gamma_j}\right) T_\alpha E_\alpha &= \prod_{j-i} \left(T_w T_{\alpha_i} T_{\alpha_j} T_{\alpha_i}^{-1} T_w^{-1}\right) T_w T_{\alpha_i} T_w^{-1} T_w E_{\alpha_i} \\ &= T_w T_{\alpha_i} \left(\prod_{j-i} T_{\alpha_j}\right) T_{\alpha_i}^{-1} T_w^{-1} T_w T_{\alpha_i} T_w^{-1} T_w E_{\alpha_i} \\ &= T_w T_{\alpha_i} \left(\prod_{j-i} T_{\alpha_j}\right) E_{\alpha_i} \\ &= E_{C\alpha} \qquad \square \end{aligned}$$

Now fix a height function h, and hence a choice of compatible simple roots Π_h, an orientation Ω_h, a reduced expression w_0 and a slice $\Gamma_h \subset \widehat{\Gamma}$. Define a root basis as follows:

(1) For $\beta \in \Gamma_h$ choose $E_\beta \in \mathfrak{g}_\beta$.
(2) Since any root is of the form $\alpha = C^k\beta$ for some k and some β, define E_α inductively using Eq. (4.5), beginning with $E_{C\beta}$ for β a source in Γ_h.

Note that for $C^h\beta = \beta$ this procedure produces another root vector $E'_\beta \in \mathfrak{g}_\beta$.

Proposition 4.4. *Let E_α, E'_α be the root vectors defined above.*

(1) *For $q = 1$, $E'_\alpha = E_\alpha$, so this procedure produces a consistent root basis in $\mathfrak{g}$.*
(2) *For $q \neq 1$, $E'_\alpha = K_\alpha^{-1} E_\alpha K_\alpha$.*

Corollary 4.5. *Let $U_q\mathfrak{g}$ denote the corresponding quantum group. For $q \neq 1$ there is a $\mathbb{Z}$-torsor of vectors $\{E_\alpha^k\}$ for each root α that are related by $E_\alpha^{k+n} = K_\alpha^{-n} E_\alpha^k K_\alpha^n$.*

Proof. To simplify notation, set $E_{\alpha_i} = E_i$, $F_{\alpha_i} = F_i$, $K_{\alpha_i} = K_i$. Then using Corollary 3.3 one obtains:

$$\begin{aligned} E'_i &= T_{w_0}(T_{w_0} E_i) \\ &= T_{w_0}(-F_{i^\vee} K_{i^\vee}) \\ &= -(T_{w_0} F_{i^\vee})(T_{w_0} K_{i^\vee}) \\ &= -\big(-K_i^{-1} E_i\big)(K_i) \\ &= K_i^{-1} E_i K_i \end{aligned}$$

This proves the second part, and to get the first part set $q = 1$ so that $K_i = 1$. $\square$

Theorem 4.6. *Let h be any height function and denote the associated simple roots and orientation by Π_h and Ω_h respectively.*

(1) *The root basis defined above is adapted to the pair (Π_h, Ω_h).*
(2) *For this choice of root basis the Lie bracket is given by*:

$$[E_\alpha, E_\beta] = \begin{cases} (-1)^{\langle\alpha,\beta\rangle} E_{\alpha+\beta} & \text{for } \alpha+\beta \in R \\ 0 & \text{for } \alpha+\beta \notin R \text{ and } \alpha \neq -\beta \end{cases}$$

Proof. Let h be the height function used to construct the root basis $\{E_\alpha\}$. By construction this basis is adapted to the pair (Π_h, Ω_h). So it is enough to check that if $\{E_\alpha\}$ is adapted to (Π, Ω), and i is a source for Ω, then $\{E_\alpha\}$ is also adapted to $(s_i\Pi, s_i\Omega)$.

Suppose that $\{E_\alpha\}$ is adapted to (Π, Ω) and that i is a source. Let $\Pi = \{\alpha_1, \dots, \alpha_r\}$. Then since i is a source and w_0 is adapted to Ω, the corresponding reduced expression for the longest element has the form $w_0 = s_i s_{i_2} \cdots s_{i_l}$. By writing $\gamma_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$, the longest element can be reexpressed as $w_0 = s_{\gamma_l} \cdots s_{\gamma_2} s_{\alpha_i}$. (Note that since i is a source, $\gamma_1 = \alpha_i$.)

Then since $\{E_\alpha\}$ is adapted it is possible to write

$$E_{\gamma_k} = T_{\gamma_{k-1}} \cdots T_{\gamma_2} T_{\alpha_i} E_{\alpha_k}$$

Now consider the pair $(s_i\Pi, s_i\Omega)$. Denote the simple roots by $\alpha'_j = s_i\alpha_j$ and the corresponding simple reflections $s'_j = s_i s_j s_i$. Then the corresponding reduced expression for the longest element is $w_0 = s'_{i_2} \cdots s'_{i_l} s_i$ and as before if $\gamma'_k = s'_{i_2} \cdots s'_{i_{k-1}}(\alpha'_{i_k})$, then $\gamma'_k = \gamma_{k+1}$ for $k+1 \neq l$ and $\gamma_l = -\alpha_i$.

Now, if $k+1 \neq l$ then

$$\begin{aligned} E'_{\gamma_k} &= E_{\gamma_{k+1}} \\ &= T_{\gamma_k} \cdots T_{\gamma_2} T_{\alpha_i} E_{\alpha_{i_{k+1}}} \\ &= T_{\gamma_k} \cdots T_{\gamma_2} E_{s_i\alpha_{i_{k+1}}} \quad \text{by Eq. (4.4)} \\ &= T_{\gamma'_{k-1}} \cdots T_{\gamma'_1} E_{\alpha'_k} \end{aligned}$$

so E'_{γ_k} is given by Eq. (4.3).

If $k+1 = l$, then

$$\begin{aligned} E'_{\gamma_l} &= E_{-\alpha_i} \\ &= T_{w_0}(E_{\alpha_i{}^\vee}) \\ &= T_{\gamma_l} \cdots T_{\gamma_2} T_{\alpha_i}(E_{\alpha_i{}^\vee}) \\ &= T'_{\gamma_{l-1}} \cdots T'_{\gamma_1} E_{s_i\alpha_i{}^\vee} \\ &= T'_{\gamma_{l-1}} \cdots T'_{\gamma_1} E_{\alpha'_i{}^\vee} \end{aligned}$$

so again E'_{γ_k} is given by Eq. (4.3). Hence $\{E_\alpha\}$ is adapted to the pair $(s_i\Pi, s_i\Omega)$.

The proof of the second part will follow from Corollary 5.4. □

Note that Proposition 4.4 and Proposition 4.6 prove the main result, Theorem 1.1.

Define $T_C = T_{\alpha_{i_1}} T_{\alpha_{i_2}} \cdots T_{\alpha_{i_r}}$, for some choice of compatible simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$, with $C = s_{i_1} s_{i_2} \cdots s_{i_r}$. Since the T_α satisfy the braid relations, the operator does not depend on the choice of compatible simple roots Π.

Proposition 4.7. *The root vectors E_α satisfy $T_C E_\alpha = E_{C\alpha}$.*

5. Ringel–Hall algebras

In this section Ringel and Peng and Xiao's approaches to constructing the Lie algebra $\mathfrak{g}$ from quiver theory is reviewed. This is then related to the construction given in the previous section. For more details on Ringel's construction see [R1,R2,DX]. For more details on Peng and Xiao's construction see [PX1] and [PX2].

Let Ω be a fixed orientation of Γ and denote by $\vec{\Gamma} = (\Gamma, \Omega)$ the corresponding quiver. Fix $\mathbb{K}$, a finite field of order p. Let $\mathrm{Rep}(\vec{\Gamma})$ be the category of representations of this quiver over the field $\mathbb{K}$, and denote by $\mathcal{K}$ its Grothendieck group. Denote by $\mathrm{Ind} \subset \mathcal{K}$ the set of classes of indecomposable objects. Then Gabriel's theorem gives an identification $\mathrm{Ind} \to R_+$ between indecomposable objects and positive roots of the corresponding root system. Moreover, if $\langle\cdot,\cdot\rangle$ is defined on $\mathcal{K}$ by $\langle X, Y\rangle = \dim \mathrm{Hom}(X,Y) - \dim \mathrm{Ext}^1(X,Y)$, then the form given by $(X,Y) = \langle X, Y\rangle + \langle Y, X\rangle$ is identified with the bilinear form on the root lattice. The form $\langle\cdot,\cdot\rangle$ is called the Euler form.

Ringel then constructed an associative algebra $(\mathcal{H}_p, *)$ as follows:

(1) As a vector space $\mathcal{H}_p$ is spanned by $[M] \in \mathcal{K}$.
(2) For objects M, N, L define $F^L_{M,N} = |\{X \subset L \mid X \simeq M \text{ and } L/X \simeq N\}|$. (Since $\mathbb{K}$ is finite, this number is well defined.)
(3) Define an operation $*$ on $\mathcal{H}_p$ by the formula $[M] * [N] = \sum_{[L]} F^L_{M,N}[L]$.

The following theorem summarizes the main results of Ringel.

Theorem 5.1. *Let $(\mathcal{H}_p, *)$ be the algebra defined above.*

(1) *For $n = (n_\alpha) \in (\mathbb{Z}_+)^{R_+}$ set $M_n = \bigoplus n_\alpha M_\alpha$ where M_α is the indecomposable corresponding to root α. Then $\{[M_n]\}$ is a PBW-type basis of the algebra $\mathcal{H}_p$, so that all structure constants $F_{[M],[N]}$ are in $\mathbb{Z}[p]$. Hence the Hall algebra can be considered with p as a formal parameter. After making the substitution $q = p^{1/2}$, $\mathcal{H}_q$ can be identified with $U_q\mathfrak{n}_+$.*
(2) *For $q = 1$, this gives an isomorphism $\Psi : U\mathfrak{n}_+ \to \mathcal{H}_1$ which is given by $E_\alpha \mapsto [M_\alpha]$, where M_α denotes the indecomposable representation of $\overrightarrow{\Gamma}$ corresponding to root α. In particular the set $\{[M_\alpha]\}$ is a root basis for $\mathfrak{n}_+$.*
(3) *In the case $q = 1$, the Lie bracket $[\cdot,\cdot]$ is given by $[M_\alpha, M_\beta] = (-1)^{\langle M_\alpha, M_\beta\rangle} M_{\alpha+\beta}$ for $\alpha + \beta \in R$. Here $\langle\cdot,\cdot\rangle$ is the Euler form.*

The polynomial $F^L_{M,N}(p)$ appearing in part (1) of the theorem is called the "Hall polynomial".

As mentioned before, Peng and Xiao extended the results of Ringel to obtain a description for all of $\mathfrak{g}$. This construction is briefly recalled here. For a more details see [PX1,PX2].

Peng and Xiao considered the "root category", $\mathcal{D} = \mathcal{D}^b(\overrightarrow{\Gamma})/T^2$, so that indecomposable objects are in bijection with all roots. If $M \in \mathrm{Rep}(\overrightarrow{\Gamma})$ is indecomposable, then considering this as a complex concentrated in degree 0, M is also indecomposable in $\mathcal{D}$. These objects correspond to positive roots, while their translates, TM, correspond to negative roots. (Up to isomorphism, this is a full description of indecomposable objects in $\mathcal{D}$.) Denote by M_α the class of indecomposable corresponding to root $\alpha \in R_+$. Peng and Xiao then constructed a Lie algebra $\mathcal{H}_\mathcal{D}$ as follows:

(1) Set $\mathcal{H}_\mathcal{D} = \mathfrak{N} \oplus \mathfrak{H}$ where $\mathfrak{H} = \mathcal{K}(\mathcal{D})$ and $\mathfrak{N}$ is the free abelian group with basis $\{u_{[M]}\}$ indexed by isomorphism classes of objects $[M]$.
(2) Let $h_M = [M] \in \mathcal{K}(\mathcal{D})$.
(3) Define a bilinear operation $[\cdot,\cdot]$ on $\mathcal{H}_\mathcal{D}$ by:
    (a) $[\mathfrak{H}, \mathfrak{H}] = 0$.
    (b) $[u_M, u_N] = \sum_{[L]}(F^L_{M,N}(1) - F^L_{N,M}(1))u_L$, for $N \neq TM$, where $F^L_{M,N}$ is the Hall polynomial.
    (c) $[u_M, u_{TM}] = \frac{h_M}{d(M)}$ where $d(M) = \dim \mathrm{End}(M)$.
    (d) $[h_M, u_N] = -(M, N)_\mathcal{D} u_N = -[u_N, h_M]$ where $(\cdot,\cdot)_\mathcal{D}$ is the symmetrized Euler form on $\mathcal{K}(\mathcal{D})$.
(4) For $\alpha \in R_+$, let $h_\alpha = h_M$ where $\dim M_\alpha = \alpha$.
(5) For $\alpha \in R_+$, let $M_\alpha = [M_\alpha]$ where $\dim M_\alpha = \alpha$.
(6) For $\alpha \in R_+$, let $M_{-\alpha} = -[TM_\alpha]$ where $\dim M_\alpha = \alpha$.

Theorem 5.2. *Let $(\mathcal{H}_\mathcal{D}, [\cdot,\cdot])$ be defined as above.*

(1) *$(\mathcal{H}_\mathcal{D}, [\cdot,\cdot])$ is a Lie algebra.*
(2) *The collection $\{M_\alpha, M_{-\alpha}\}_{\alpha\in R_+}$ defined above is a root basis for $\mathcal{H}_\mathcal{D}$.*
(3) *The map given by $E_\alpha \mapsto M_\alpha$, $F_\alpha \mapsto M_{-\alpha}$ and $H_\alpha \mapsto h_\alpha$ for $\alpha \in R_+$ induces an isomorphism of Lie algebras $\mathfrak{g} \to \mathcal{H}_\mathcal{D}$. Hence $\mathcal{H}_\mathcal{D}$ can be identified with the $\mathbb{Z}$-form of $\mathfrak{g}$.*

For details see [PX1, Section 4].

Recall that given a height function h, there is a corresponding set of simple roots Π_h and a polarization $R = R^h_+ \cup R^h_-$. Let E_α be the root vectors defined in Section 4. Define a triangular decomposition $\mathfrak{g} = \mathfrak{n}^h_- \oplus \mathfrak{h} \oplus \mathfrak{n}^h_+$ by setting $\mathfrak{n}^h_\pm = \langle E_\alpha\rangle_{\alpha\in R^h_\pm}$.

A height function h also gives an orientation Ω_h of Γ and hence a quiver $\overrightarrow{\Gamma} = (\Gamma, \Omega_h)$. As above, denote by $\mathcal{K}$ the corresponding Grothendieck group, and by $\mathrm{Ind} \subset \mathcal{K}$ the set of indecomposable classes in $\mathcal{K}$. Then there is a bijection $R^h_+ \to \mathrm{Ind}$, given by $\alpha \mapsto [M_\alpha]$.

Proposition 5.3. *Let h be a height function. Then the identification $R^h_+ \to \mathrm{Ind}$ in $\mathrm{Rep}(\overrightarrow{\Gamma})$ induces an isomorphism $U\mathfrak{n}^h_+ \to \mathcal{H}_1$ given by $E_\alpha \mapsto [M_\alpha]$.*

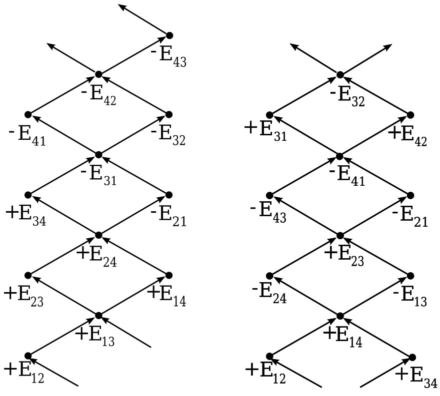

Fig. 4. Two different root bases for $\mathfrak{sl}_4$ coming from different choices of C. The case $C = (1234)$ is shown in the figure to the left. The case $C = (1243)$ is shown in the figure to the right. For each vertex in $\widehat{\Gamma}$ the corresponding root vector E_α is shown in terms of the matrix units E_{ij}. Recall that $\widehat{\Gamma}$ is periodic, so that arrows leaving the top level are identified with the incoming arrows on the bottom level.

Moreover, the identification $R \to \mathrm{Ind}(\mathcal{D})$ in the root category $\mathcal{D}$ gives an isomorphism $\mathfrak{g} - \mathfrak{h} \to \mathcal{H}_\mathcal{D} - \mathfrak{H}$, given by $E_\alpha \mapsto [M_\alpha]$, $E_{-\alpha} \mapsto -[TM_\alpha]$ for $\alpha \in R^h_+$.

Corollary 5.4. *The Lie algebra $\mathfrak{g}$ can be realized combinatorially in terms of $\widehat{\Gamma}$: It has root basis E_α for $\alpha \in \widehat{\Gamma}$ and Lie bracket given by*

$$[E_\alpha, E_\beta] = \begin{cases} (-1)^{\langle\alpha,\beta\rangle} E_{\alpha+\beta} & \text{for } \alpha+\beta \in R \\ 0 & \text{for } \alpha+\beta \notin R \text{ and } \alpha \neq -\beta \end{cases} \tag{5.1}$$

Proof. The only thing to be checked is that in terms of the E_α constructed in Section 4, the structure constants of the Lie bracket are given by Eq. (5.1). For $\alpha, \beta \in R$ with $\alpha \neq -\beta$ there is a choice of compatible simple roots Π so that $\alpha, \beta \in R^\Pi_+$. Let h be the corresponding height function. Then by Proposition 5.3 the identification $U\mathfrak{n}^h_+ \simeq \mathcal{H}_1$ gives that

$$[E_\alpha, E_\beta] = [M_\alpha, M_\beta] = (-1)^{\langle\alpha,\beta\rangle} M_{\alpha+\beta} = (-1)^{\langle\alpha,\beta\rangle} E_{\alpha+\beta} \qquad \square$$

Remark 5.5. Note that the "Euler cocylce" $(-1)^{\langle\cdot,\cdot\rangle}$ defines a cohomologous cocycle, and hence the same extension, as in the construction of $\mathfrak{g}$ given in [FLM].

Example 5.6. Consider the case $\Gamma = A_3$, so that $\mathfrak{g} = \mathfrak{sl}_4$. Let $\mathfrak{h}$ be the diagonal matrices. Then the roots are $\alpha = e_i - e_j$ for $i \neq j$, where $e_k(h) = h_{kk}$ for $h \in \mathfrak{h}$. The root space corresponding to root $e_i - e_j$ is $\mathbb{C}E_{ij}$, where E_{ij} is the corresponding matrix unit. For two different choices of Coxeter element C, two different root bases are shown in Fig. 4. In each case the Lie bracket is then given by Eq. (5.1) and the form $\langle\cdot,\cdot\rangle$ can be computed explicitly in terms of $\widehat{\Gamma}$.

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